Solution 9

In the following the Initial Value Problem (IVP) refers to $x' = f(t,x),\ x(t_0) = x_0$, where f satisfies the Lipschitz condition in some rectangle containing (t_0, x_0) in its interior, see Notes for details.

1. Solve the (IVP) for $f(t,x) = \alpha t(1+x^2), \alpha > 0,\ t_0 = 0$, and discuss how the interval of existence changes as α and x_0 vary.

   Solution. The solution is given by

   $$x(t) = \tan(\tan^{-1} x_0 + \alpha t^2/2) \ ,$$

   where the tangent function is chosen so that $\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$. The (maximal) interval of existence is $(-a, a)$ where

   $$a = \frac{1}{\alpha}(\pi - 2\tan^{-1} x_0) \ .$$

   We see that for fixed α, the interval shrinks as x_0 increases, and for fixed x_0, it shrinks too as α increases. The maximal interval of existence depends on f, t_0 and x_0 in a complicated manner.

2. Let x be a solution to the IVP on (c, d), a subinterval of (a, b). Show that it extends to be a solution on $[c, d]$.

   Solution. Pick any sequence $t_n \uparrow d$. The sequence $\{x(t_n)\}$ belongs to $[\alpha, \beta]$ and hence is bounded. (Here we take $R = [a, b] \times [\alpha, \beta]$ as usual.) There is a subsequence $\{s_k\}$ of $\{t_n\}$ so that $x(s_k)$ converges to some point x_1. We claim $\lim_{t\uparrow d} x(t) = x_1$. For, we have

   $$|x(t) - x(s_k)| = |\int_{s_k}^{t} f(s, x(s))\, ds| \le M|t - s_k| \ .$$

   By letting $k \to \infty$, we get $|x(t) - x_1| \le M|t - d|$, from which we deduce $\lim_{t\uparrow d} x(t) = x_1$. Now, we can extend x to up to d by defining $x(d) = x_1$ so that it is continuous up to d. Moreover, letting $k \to \infty$ in

   $$x(s_k) - x(t) = \int_t^{s_k} f(s, x(s))\, ds \ ,$$

   we get

   $$x(d) - x(t) = \int_t^{d} f(s, x(s))\, ds \ .$$

   Since x is continuous at d, by the Second Fundamental Theorem

   $$x'(d) = \lim_{t\uparrow d} \frac{f(d) - x(t)}{d - t} = f(d, x(d)).$$

   Hence x is differentiable at d (more precisely, left derivative exists) and satisfies the differential equation.

3. Let $x_i, i = 1, 2$, be two solutions to the same IVP on the subinterval I_i of $[a, b]$ satisfying $\alpha < x_i(t) < \beta$. Show that x_1 is equal to x_2 on $I_1 \cap I_2$.

   Solution. Let $I = I_1 \cap I_2$. For $i = 1, 2$, we have

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds \ , \quad t \in I \ .$$

By subtracting, as $x_1(t_0) = x_2(t_0)$,

$$\begin{aligned} |x_1(t) - x_2(t)| &= \left| \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_2)|\, ds \right| \\ &\le L \left| \int_{t_0}^{t} |x_1(s) - x_2(s)|\, ds \right| \ . \end{aligned}$$

Let us take $t > t_0$. (The case $t < t_0$ can be handled similarly.) The function

$$H(t) \equiv \int_{t_0}^{t} |x_1(s) - x_2(s)|\, ds$$

satisfies the differential inequality

$$H'(t) \le LH(t) \ , \quad t \in I^+, \quad I^+ = I \cap \{t > t_0\}.$$

It satisfies $H(t_0) = 0$ and is always increasing. Moreover, it vanishes on I^+ if and only if x_1 coincides with x_2 on I^+. To show that H vanishes, we add an $\varepsilon > 0$ to the right hand side of this differential inequality to get $H' \le L(H + \varepsilon)$. Writing it as $(\log(H + \varepsilon))' \le L$, and integrating it to get

$$\log(H(t) + \varepsilon) - \log \varepsilon \le L(t - t_0) \ ,$$

or

$$H(t) \le \varepsilon e^{L(t-t_0)}, \quad t \in I^+ \ .$$

Now the desired conclusion follows by letting $\varepsilon \to 0$.

Note. This problem is essentially Proposition 3.12 in the revised Chapter 3.

4. Optional. Deduce Picard-Lindelöf Theorem based on the ideas of perturbation of identity. Hint: Take a particular

$$y = \int_{t_0}^{t} f(t, x_0) dt$$

in the relation $x + \Psi(x) = y$.

Solution. Write the integral form of (IVP) as

$$x(t) - x_0 - \int_{t_0}^{t} (f(s, x(s)) - f(s, x_0)) ds = \int_{t_0}^{t} f(s, x_0) ds \ .$$

Define $Tx(t) = \Psi(x) + y$, where

$$\Psi(x) = -x_0 - \int_{t_0}^{t} (f(s, x(s)) - f(s, x_0)) ds \ .$$

Let

$$X = \{x \in C[t_0 - a', t_0 + a'] : |x(t) - x_0| \le b\}$$

where $a' = \min\{a, b/M, 1/L\}$ as before. We first claim, when $a' \le b/M$, T maps X to itself. Indeed,

$$|Tx(t) - x_0| = |\int_{t_0}^{t} f(s, x(s))ds| \le M|t| \le b ,$$

by our choice. Next, claim T is a contraction on X. We have

$$|Tx_1(t) - Tx_2(t)| = |\Psi(x_1)(t) - \Psi(x_2)(t)| = \left| \int_{t_0}^{t} (f(s, x_1(s) - f(s, x_2(s)) \, ds \right| \le L|t| \le a'$$

by our choice. Now, apply Contraction Mapping Principle to T on X to get a unique fixed point. It is the solution of our (IVP).

5. Show that the solution to IVP belongs to C^{k+1} (as long as it exists) provided $f \in C^k$ for $k \ge 1$. In particular, $y \in C^\infty$ provided $f \in C^\infty$.

   Solution. It is an elementary fact and easy to show that the composition of two C^k-functions is again C^k. Now, from (1) we see that y is C^1 if the RHS, that is, $f(x, y(x))$ is continuous. By induction, assuming now y is C^{k+1} when f is C^k. When f is C^{k+1}, it is also C^k and so by induction hypothesis y is C^{k+1}. The RHS of (1) is the composition of twonC^{k+1}-functions and hence is also C^{k+1}. It shows that the LHS y' is C^{k+1}, that is, $y \in C^{k+2}$, done.

6. Consider the IVP for second order equation:

$$x'' = f(t, x, x'), \quad x(t_0) = x_0, \; x'(t_0) = x_1 ,$$

   where $f \in C(R), R = [a, b] \times [\alpha, \beta] \times [\gamma, \delta]$. Assume that f satisfies the Lipschitz condition

$$|f(t, x, x') - f(t, y, y')| \le L(|x - y| + |x' - y'|) , \quad (t, x, x'), (t, y, y') \in R .$$

   Show that the IVP admits a unique solution in $(t_0 - \rho, t_0 + \rho)$ for some $\rho > 0$ by carrying out the following steps.

   (a) Show that the IVP is equivalent to solving

$$x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{s} f(r, x(r), x'(r)) \, drds .$$

   (b) Verify the space $C^1[a, b]$ is complete under the norm

$$\|x\|_1 = \|x\|_\infty + \|x'\|_\infty .$$

   (c) Apply the Contraction Mapping Principle in a closed subset of $(C^1[a, b], \| \cdot \|_1)$.

   Solution. (a) As the first order case, except now we integrate one more time.

   (b) Let $\{x_n\}$ be a Cauchy sequence in this normed space. It means that both $\{x_n\}$ and $\{x'_n\}$ are Cauchy sequence in supnorm. By the completeness of $C[a, b]$ in supnorm, there are $x, z \in C[a, b]$ such that x_n and x'_n converge to x and z uniformly. From the defining relation

$$x_n(t) - x_n(s) = \int_{s}^{t} x'_n(r) \, dr ,$$

we pass limit to get

$$x(t) - x(s) = \int_s^t z(r)\,dr\ ,$$

which shows that $z = x'$, so $\{x_n\}$ converges in the norm $\|\cdot\|_1$.

(c) It is routine to verify, for each small $\rho > 0$, the set

$$X = \{x \in C^1[t_0 - \rho, t_0 + \rho] :\ x(t) \in [\alpha, \beta], x'(t) \in [\gamma, \delta]\}$$

is a closed subset in $C^1[a, b]$ so it is also complete under $\|\cdot\|_1$. As in the first order case, we define

$$(Tx)(t) = x_0 + x_1(t - t_0) + \int_{t_0}^t \int_{t_0}^s f(r, x(r), x'(r))\,drds\ ,$$

and verify that when δ is small, it is a contraction from X to X and hence admits a fixed point.

7. Show that there exists a unique solution h to the integral equation

$$h(x) = 1 + \frac{1}{\pi}\int_{-1}^1 \frac{1}{1 + (x - y)^2} h(y)dy,$$

in $C[-1, 1]$. Also show that h is non-negative.

Solution. Let $X = C[-1, 1]$ be the complete metric space we work on and set

$$(Th)(x) = 1 + \frac{1}{\pi}\int_{-1}^1 \frac{1}{1 + (x - y)^2} h(y)dy.$$

It is easy to check that T is continuous on X. For $h_2, h_1 \in C[-1, 1]$, we have

$$\begin{aligned} |Th_2(x) - Th_1(x)| &= \left| \frac{1}{\pi}\int_{-1}^1 \frac{1}{1 + (x - y)^2}(h_2(y) - h_1(y))dy \right| \\ &\le \frac{2}{\pi}\|h_2 - h_2\|_\infty, \quad \forall x \in [-1, 1]. \end{aligned}$$

Hence T is a contraction on $C[-1, 1]$, and a fixed point is ensured by Banach's Fixed Point Theorem.

Next we show that the fixed point h is non-negative. Notice that

$$\frac{1}{\pi}\int_{-1}^1 \frac{1}{1 + (x - y)^2}dy = \frac{1}{\pi}[\arctan(1 - x) + \arctan(1 + x)] \le \frac{1}{2}, \quad x \in [-1, 1].$$

From the def of h we have

$$\|h\|_\infty \le 1 + \frac{1}{2}\|h\|_\infty,$$

which implies $\|h\|_\infty \le 2$. It follows that

$$h(x) \ge 1 - \frac{1}{\pi}\int_{-1}^1 \frac{1}{1 + (x - y)^2}\|h\|_\infty dy \ge 1 - \frac{1}{2} \times 2 \ge 0,$$

h is non-negative.

An alternate approach. We work on the space $Y = \{h \in C[-1, 1] :\ h(x) \ge 0, \forall x\}$. From the definition of T, it is clear that T maps Y to Y. Since Y is easily shown to be a closed set in $C[-1, 1]$ (hence complete), we apply the Contraction Mapping Principle directly to get a non-negative solution.